

# UNIFORM PERFECTNESS, (POWER) QUASIMÖBIUS MAPS AND (POWER) QUASISYMMETRIC MAPS IN QUASI-METRIC SPACES

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**ABSTRACT.** The aim of this paper is to investigate the uniform perfectness in quasi-metric spaces. One of the main results in this paper is the invariant property of uniform perfectness under quasimöbius maps in quasi-metric spaces. This result is built on the equivalence of uniform perfectness with homogeneous density,  $\sigma$ -density etc. The second main result is the relationships of uniform perfectness with (power) quasimöbius maps and (power) quasisymmetric maps. In the end, two applications of the first main result are given.

## 1. INTRODUCTION AND MAIN RESULTS

We start with the definition of quasi-metric spaces.

**Definition 1.1.** For a given set  $Z$  and a constant  $K \geq 1$ ,

- (1) a function  $\rho : Z \times Z \rightarrow [0, +\infty)$  is said to be  $K$ -quasi-metric
  - (a) if for all  $x$  and  $y$  in  $Z$ ,  $\rho(x, y) \geq 0$ , and  $\rho(x, y) = 0$  if and only if  $x = y$ ;
  - (b)  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in Z$ ;
  - (c)  $\rho(x, z) \leq K(\rho(x, y) \vee \rho(y, z))$  for all  $x, y, z \in Z$ .
- (2) the pair  $(Z, \rho)$  is said to be a  $K$ -quasi-metric space if the function  $\rho : Z \times Z \rightarrow [0, +\infty)$  is  $K$ -quasi-metric with  $K \geq 1$ . In the following, we always say that  $K$  is the quasi-metric coefficient of  $(Z, \rho)$ .

Here and hereafter, we use the notations:  $r \vee s$  and  $r \wedge s$  for numbers  $r, s$  in  $\mathbb{R}$ , where

$$r \vee s = \max\{r, s\} \quad \text{and} \quad r \wedge s = \min\{r, s\}.$$

Obviously, if  $(Z, \rho)$  is  $K_1$ -quasi-metric, it must be  $K_2$ -quasi-metric for any  $K_2 \geq K_1$ . Hence, in the following, the quasi-metric coefficients of all quasi-metric spaces are always assumed to be  $K$  with  $K > 1$ .

Every quasi-metric  $\rho$  defines a uniform structure on  $Z$ . The balls  $\mathbb{B}(x, r) = \{y \in Z : \rho(x, y) < r\}$  ( $r > 0$ ) form a basis of neighbourhoods of  $x$  for the topology induced by the uniformity on  $Z$ . This topology is a metric one since the uniform

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structure associated to  $\rho$  has a countable basis. We shall refer to this topology as the  $\rho$ -topology of  $Z$  (cf. [18]).

Quasi-metric spaces are not quite as well behaved as metric spaces. For instance, the unit disk in the plane with a power (which is less than one) of the Euclidean metric is clearly a quasi-metric space and one can check that there are no rectifiable curves in this space except the constant curves. The following useful result on the relationship between quasi-metric spaces and metric spaces easily follows from [7, Proposition 2.2.5].

**Lemma 1.1.** *Let  $(X, \rho)$  be a  $K$ -quasi-metric space. If there exists some constant  $0 < \varepsilon \leq 1$  such that  $K^\varepsilon \leq 2$ , then there is a metric  $d_\varepsilon$  on  $X$  such that*

$$\frac{1}{4}\rho^\varepsilon(z_1, z_2) \leq d_\varepsilon(z_1, z_2) \leq \rho^\varepsilon(z_1, z_2)$$

for all  $z_1, z_2 \in X$ .

For more properties concerning quasi-metric spaces, see [2, 4, 6, 7, 8, 9, 10, 11, 16, 18, 23, 29, 30, 31] etc.

To state our results, we need the definition of uniform perfectness in quasi-metric spaces.

**Definition 1.2.** A quasi-metric space  $(Z, \rho)$  is called *uniformly perfect* if there is a constant  $\mu \in (0, 1)$  such that for each  $x \in Z$  and every  $r > 0$ , the set  $\mathbb{B}(x, r) \setminus \mathbb{B}(x, \mu r) \neq \emptyset$  provided that  $Z \setminus \mathbb{B}(x, r) \neq \emptyset$ .

Uniform perfectness is a weaker condition than connectedness. Connected spaces are uniformly perfect, and those with isolated points are not. Many disconnected fractals such as the Cantor ternary set, Julia sets and the limit set of a nonelementary, finitely generated Kleinian group of  $\mathbb{R}^n$  are uniformly perfect [15, 19, 24]. In particular, uniform perfectness has provided a useful tool in the solution of many problems in the theory of complex analysis. It seems that this notion first appeared in [21], but almost at the same time, an equivalent concept appeared in [26] under the name “homogeneous density sets” in the setting of metric spaces. It is worth mentioning that Buyalo and Schroeder recently established the quasisymmetric and quasimöbius extension theorems for visual geodesic hyperbolic spaces which possess uniformly perfect boundaries at infinity [7, Chapter 7]. There are many references in literature about uniform perfectness; see [1, 7, 9, 11, 13, 14, 15, 21, 22] etc. Among others, the authors in [15] proved several characterizations of uniform perfectness of subsets in  $\mathbb{R}^n$  (see [15, Theorem 4.5]).

As the first goal of this paper, we shall discuss the relationship among uniform perfectness, homogeneous density,  $\sigma$ -density etc in the setting of quasi-metric spaces. Our result is as follows, which plays a key role in the proof of the main results. (Note that other notions appearing in the following results will be introduced in the body of this paper.)

**Theorem 1.1.** *Suppose  $(Z, \rho)$  is a quasi-metric space. Then the following are quantitatively equivalent.*

- (1)  $(Z, \rho)$  is  $\mu$ -uniformly perfect;

- (2)  $(Z, \rho)$  is  $(\lambda_1, \lambda_2)$ -HD;
- (3)  $(Z, \rho)$  is  $\sigma$ -dense;
- (4) There are numbers  $\mu_1$  and  $\mu_2$  such that  $0 < \mu_1 \leq \mu_2 < 1$  and for any triple  $(a, c, d)$  in  $Z$ , there is a point  $x \in Z$  satisfying  $\mu_1 \leq r(a, x, c, d) \leq \mu_2$ .

Here, we make the following notational convention: Suppose  $A$  denotes a condition with data  $v$  and  $B$  another condition with data  $v'$ . We say that  $A$  implies  $B$  quantitatively if  $A$  implies  $B$  so that  $v'$  depends only on  $v$ . If  $A$  and  $B$  imply each other quantitatively, then we say that they are quantitatively equivalent.

**Remark 1.1.** In metric spaces, the equivalence between (1) and (4) (resp. (2) and (3), (3) and (4)) coincides with [11, Lemma 11.7] (resp. [14, Lemma 3.1], [1, Remark 3.3])

Quasisymmetric maps made their first official appearance in the 1956 paper by Beurling and Ahlfors [3]. This concept was later promoted by Tukia and Väisälä, who introduced and studied quasisymmetric maps between metric spaces [26]. In [12], Heinonen and Koskela established the equivalence of quasiconformal and quasisymmetric maps in Ahlfors regular metric spaces that satisfy certain bounds in geometry. In [29], Väisälä introduced the term quasimöbius maps in metric spaces and obtained the close connections among quasimöbius maps, quasiconformal maps and quasisymmetric maps. See [1, 5, 6, 7, 11, 26, 27, 28] for more background materials in this line. Further, in [26], Tukia and Väisälä proved that every quasisymmetric map in homogeneous spaces is power quasisymmetric. As the second goal of this paper, we shall investigate the relationships of the uniform perfectness with (power) quasisymmetric maps and (power) quasimöbius maps, respectively. Our results are the following, which are the main results in this paper.

**Theorem 1.2.** *Suppose  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  is a quasimöbius map between quasi-metric spaces  $(Z_1, \rho_1)$  and  $(Z_2, \rho_2)$ . Then  $(Z_1, \rho_1)$  is uniformly perfect if and only if  $(Z_2, \rho_2)$  is uniformly perfect, quantitatively.*

We remark that Theorem 1.2 coincides with [15, Corollary 4.6] in the setting of  $\mathbb{R}^n$ .

**Theorem 1.3.** *Suppose  $(Z, \rho)$  is a quasi-metric space, if  $(Z, \rho)$  has no isolated points, then*

- (1)  $(Z, \rho)$  is uniformly perfect if and only if every quasisymmetric map of  $(Z, \rho)$  to a quasi-metric space is power quasisymmetric, quantitatively;
- (2)  $(Z, \rho)$  is uniformly perfect if and only if every quasimöbius map of  $(Z, \rho)$  to a quasi-metric space is power quasimöbius, quantitatively.

**Remark 1.2.** (1) In metric spaces, Theorem 1.3(1) is the same as [25, Theorems 4.13 and 6.20]. (2) Aseev and Trotsenko proved that, in  $\mathbb{R}^n$ , if  $(Z, \rho)$  is  $\sigma$ -dense, then every quasimöbius map of  $(Z, \rho)$  is power quasimöbius by applying the conformal moduli of families of curves (see [1, Theorem 4.1]). Obviously, the method of proof in [1] is invalid in metric spaces. So, even in  $\mathbb{R}^n$ , the method of proof of Theorem 1.3(2) is new.

Recently, Meyer studied the relationship between Gromov hyperbolic spaces and their boundaries at infinity. He proved the invariant property of the uniform perfectness with respect to the inversions in quasi-metric spaces (see [20, Theorem 7.1]). This result is one of the main results in [20], whose proof is lengthy. As an application of Theorem 1.2, we shall give a different proof to [20, Theorem 7.1] (see Theorem 6.1 below). Also, we shall discuss the uniform perfectness of a complete quasi-metric space and the corresponding boundary of its hyperbolic approximation by applying Theorem 1.2 (see Theorem 6.2 below).

The organization of this paper is as follows. In the second section, we shall introduce some necessary concepts and discuss the condition in quasi-metric spaces under which quasimöbius maps and quasisymmetric maps are the same. In the third section, some concepts will be introduced and the equivalence of uniform perfectness with homogeneous density,  $\sigma$ -density etc will be proved. The invariant property of uniform perfectness with respect to quasimöbius maps will be shown in the forth section, and in the fifth section, relationships among (power) quasisymmetric maps, (power) quasimöbius maps and uniform perfectness will be established. In the last section, some applications of the invariant property of uniform perfectness with respect to quasimöbius maps, i.e. Theorem 1.2, will be given.

## 2. QUASIMÖBIUS MAPS AND QUASISYMMETRIC MAPS IN QUASI-METRIC SPACES

In this section, we shall introduce some necessary notations and concepts, and prove several basic results. The main result in this section is Theorem 2.1, which concerns the condition under which power quasisymmetric maps and power quasimöbius maps are the same.

### 2.1. Cross ratios.

For four points  $a, b, c, d$  in a quasi-metric space  $(Z, \rho)$ , its *cross ratio* is defined by the number

$$r(a, b, c, d) = \frac{\rho(a, c)\rho(b, d)}{\rho(a, b)\rho(c, d)}.$$

Then we have

**Proposition 2.1.** (1) For  $a, b, c$  and  $d$  in  $(Z, \rho)$ ,

$$r(a, b, c, d) = \frac{1}{r(b, d, a, c)};$$

(2) For  $a, b, c, d$  and  $z$  in  $(Z, \rho)$ ,

$$r(a, b, c, d) = r(a, b, z, d)r(a, z, c, d).$$

In [5], Bonk and Kleiner introduced the following useful notation:

$$\langle a, b, c, d \rangle = \frac{\rho(a, c) \wedge \rho(b, d)}{\rho(a, b) \wedge \rho(c, d)}.$$

Bonk and Kleiner established a relation between  $r(a, b, c, d)$  and  $\langle a, b, c, d \rangle$  in the setting of metric spaces (see [5, Lemma 3.3]). The following result shows that this relation also holds in quasi-metric spaces.

**Lemma 2.1.** *For any  $a, b, c, d$  in  $(Z, \rho)$ , we have*

- (1)  $\frac{1}{\theta_K(1/r(a, b, c, d))} \leq \langle a, b, c, d \rangle \leq \theta_K(r(a, b, c, d));$
- (2)  $\theta_K^{-1}(\langle a, b, c, d \rangle) \leq r(a, b, c, d) \leq \frac{1}{\theta_K^{-1}(1/\langle a, b, c, d \rangle)},$

where  $\theta_K(t) = K^2(t \vee \sqrt{t})$ . (Here, we recall that  $K$  denotes the coefficient of the quasi-metric space  $(Z, \rho)$ .)

*Proof.* Obviously, we only need to prove (1) in the lemma. For the proof, we let

$$\langle a, b, c, d \rangle = s \quad \text{and} \quad r(a, b, c, d) = t.$$

Without loss of generality, we may assume  $\rho(a, c) \leq \rho(b, d)$ . Then we have

$$\begin{aligned} \rho(a, b) &\leq K(\rho(a, c) \vee \rho(c, b)) \leq K^2(\rho(a, c) \vee \rho(c, d) \vee \rho(d, b)) \\ &= K^2(\rho(c, d) \vee \rho(d, b)), \end{aligned}$$

and similarly,

$$\rho(c, d) \leq K^2(\rho(a, b) \vee \rho(d, b)).$$

The combination of these two estimates leads to

$$\rho(a, b) \vee \rho(c, d) \leq K^2((\rho(a, b) \wedge \rho(c, d)) \vee \rho(b, d)) \leq K^2(1 \vee \frac{1}{s})\rho(d, b),$$

and so we get

$$t = r(a, b, c, d) = \frac{\rho(a, c)\rho(b, d)}{(\rho(a, b) \wedge \rho(c, d))(\rho(a, b) \vee \rho(c, d))} \geq \frac{s}{K^2(1 \vee \frac{1}{s})},$$

which implies

$$s \leq \theta_K(t) = K^2(t \vee \sqrt{t}).$$

Hence the right side inequality in (1) holds.

By Proposition 2.1, we see that the left side inequality in (1) easily follows from the right side one, and so the proof of the lemma is complete.  $\square$

## 2.2. Quasisymmetric maps and quasimöbius maps.

**Definition 2.1.** Suppose  $\eta$  and  $\theta$  are homeomorphisms from  $[0, \infty)$  to  $[0, \infty)$ . A homeomorphism  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  is said to be

- (1) (a)  $\eta$ -quasisymmetric if  $\rho_1(x, a) \leq t\rho_1(x, b)$  implies

$$\rho_2(x', a') \leq \eta(t)\rho_2(x', b')$$

for all  $a, b, x$  in  $(Z_1, \rho_1)$  and  $t \geq 0$ , where primes mean the images of points under  $f$ , for example,  $x' = f(x)$  etc;

- (b) power quasisymmetric if it is  $\eta$ -quasisymmetric, where  $\eta$  has the form

$$\eta(t) = M_1(t^{1/\alpha} \vee t^\alpha)$$

for some constants  $\alpha \geq 1$  and  $M_1 \geq 1$ .

- (2) (a)  $\theta$ -quasimöbius if  $r(a, b, c, d) \leq t$  implies

$$r(a', b', c', d') \leq \theta(t)$$

for all  $a, b, c, d$  in  $(Z_1, \rho_1)$  and  $t \geq 0$ ;

(b) *power quasimöbius* if it is  $\theta$ -quasimöbius, where  $\theta$  has the form

$$\theta(t) = M_2(t^{1/\beta} \vee t^\beta)$$

for some constants  $\beta \geq 1$  and  $M_2 \geq 1$ .

As a direct consequence of Lemma 2.1, we have the following two results.

**Lemma 2.2.** *Suppose  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  is a homeomorphism between two quasi-metric spaces.*

- (1) *If  $f$  is  $\eta$ -quasisymmetric, then it is  $\theta$ -quasimöbius, where  $\theta(t) = \frac{1}{\theta_K^{-1}(\frac{1}{\eta \circ \theta_K(t)})}$  and  $\theta_K$  is from Lemma 2.1.*
- (2) *If  $f$  is a power quasisymmetric map with its control function  $\eta(t) = M(t^\alpha \vee t^{\frac{1}{\alpha}})$ , where  $M \geq 1$  and  $\alpha \geq 1$ , then it is power quasimöbius with its control function  $\theta(t) = M^2 K^{4(1+\alpha)}(t^{2\alpha} \vee t^{\frac{1}{2\alpha}})$ .*

We remark that Lemma 2.2(1) is a generalization of [29, Theorem 3.2] in the setting of quasi-metric spaces.

Next, we consider the converse of Lemma 2.2(2) in the setting of bounded quasi-metric spaces. We shall discuss the converse of Lemma 2.2(1) elsewhere. We start with the introduction of the following condition (see [29]).

**Definition 2.2.** Suppose both  $(Z_1, \rho_1)$  and  $(Z_2, \rho_2)$  are bounded quasi-metric spaces. Let  $\lambda \geq 1$  be a constant. A homeomorphism  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  is said to satisfy the  $\lambda$ -three-point condition if there are points  $z_1, z_2, z_3$  in  $(Z_1, \rho_1)$  such that

$$\rho_1(z_i, z_j) \geq \frac{1}{\lambda} \text{diam}(Z_1) \quad \text{and} \quad \rho_2(z'_i, z'_j) \geq \frac{1}{\lambda} \text{diam}(Z_2)$$

for all  $i \neq j \in \{1, 2, 3\}$ , where “diam” means “diameter”.

**Theorem 2.1.** *Suppose that both  $(Z_1, \rho_1)$  and  $(Z_2, \rho_2)$  are bounded quasi-metric spaces and that  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  satisfies the  $\lambda$ -three-point condition. Then  $f$  is power quasisymmetric if and only if it is power quasimöbius, quantitatively.*

*Proof.* The necessity of the theorem obviously follows from Lemma 2.2(2). In the following, we prove the sufficiency. Let  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  be a power quasimöbius map between two bounded quasi-metric spaces, which satisfies the  $\lambda$ -three-point condition for some constant  $\lambda \geq 1$  and points  $z_1, z_2, z_3 \in Z_1$ . We assume that the control function of  $f$  is

$$\theta(t) = M(t^{1/\beta} \vee t^\beta)$$

for some constants  $M \geq 1$  and  $\beta \geq 1$ .

To prove the power quasisymmetry of  $f$ , let  $x, a, b$  be any three points in  $(Z_1, \rho_1)$  with  $\rho_1(x, a) = t\rho_1(x, b)$  with  $t \geq 0$ . Then we shall show that

$$\rho_2(x', a') \leq \eta(t)\rho_2(x', b'),$$

where  $\eta(t) = K^{3+6\beta} M(2\lambda)^{1+2\beta} (t^{1/(2\beta)} \vee t^{2\beta})$ .

It follows from the  $\lambda$ -three-point condition that for any  $w \in Z_1$ , there are  $i \neq j \in \{1, 2, 3\}$  such that

$$\rho_1(w, z_i) \wedge \rho_1(w, z_j) \geq \frac{\text{diam}(Z_1)}{2K\lambda}.$$

Similarly, for any  $u' \in Z_2$ , there exist  $m \neq n \in \{1, 2, 3\}$  such that

$$\rho_2(u', z'_m) \wedge \rho_2(u', z'_n) \geq \frac{\text{diam}(Z_2)}{2K\lambda}.$$

Therefore, there must exist  $z_i \in \{z_1, z_2, z_3\}$  such that

$$\rho_1(a, z_i) \geq \frac{\text{diam}(Z_1)}{2K\lambda} \quad \text{and} \quad \rho_2(b', z'_i) \geq \frac{\text{diam}(Z_2)}{2K\lambda}.$$

Thus

$$\rho_1(a, z_i) \wedge \rho_1(x, b) \geq \frac{\rho_1(x, b)}{2K\lambda} \quad \text{and} \quad \rho_2(b', z'_i) \wedge \rho_2(x', a') \geq \frac{\rho_2(x', a')}{2K\lambda},$$

from which we deduce that

$$(2.1) \quad \langle x, b, a, z_i \rangle \leq 2K\lambda \frac{\rho_1(x, b)}{\rho_1(x, b)} \quad \text{and} \quad \langle x', b', a', z'_i \rangle \geq \frac{\rho_2(x', a')}{2K\lambda \rho_2(x', b')}.$$

On the other hand, since  $f$  is power quasimöbius with its control function  $\theta$ , we see from Lemma 2.1 that

$$\langle x', b', a', z'_i \rangle \leq \theta'(\langle x, b, a, z_i \rangle),$$

where  $\theta'(t) = \theta_K \circ \theta\left(\frac{1}{\theta_K^{-1}(1/t)}\right)$  and  $\theta_K$  is from Lemma 2.1. Then we deduce from (2.1) that

$$\frac{\rho_2(x', a')}{\rho_2(x', b')} \leq 2K\lambda \theta'(\langle x, b, a, z_i \rangle) \leq 2K\lambda \theta'\left(2K\lambda \frac{\rho_1(x, a)}{\rho_1(x, b)}\right).$$

By taking  $\eta(t) = K^{3+6\beta} M(2\lambda)^{1+2\beta} (t^{1/(2\beta)} \vee t^{2\beta})$ , we see from elementary computations that

$$\frac{\rho_2(x', a')}{\rho_2(x', b')} \leq \eta\left(\frac{\rho_1(x, a)}{\rho_1(x, b)}\right).$$

Hence the proof of the theorem is complete.  $\square$

**Lemma 2.3.** *Suppose  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  and  $g : (Z_2, \rho_2) \rightarrow (Z_3, \rho_3)$  are homeomorphisms.*

- (1) *If  $f$  is  $\theta_1$ -quasimöbius and  $g$  is  $\theta_2$ -quasimöbius, then  $g \circ f$  is  $\theta$ -quasimöbius, where  $\theta = \theta_2 \circ \theta_1$ ;*
- (2) *If  $f$  is  $\theta$ -quasimöbius and  $g$  is  $\eta$ -quasisymmetric, then  $g \circ f$  is  $\theta_1$ -quasimöbius, where  $\theta_1(t) = \frac{1}{\theta_K^{-1}\left(\frac{1}{\eta \circ \theta_K \circ \theta(t)}\right)}$ ;*
- (3) *If  $f$  is power quasimöbius and  $g$  is power quasisymmetric (or power quasimöbius), then  $g \circ f$  is power quasimöbius, quantitatively.*



### 3. UNIFORM PERFECTNESS, HOMOGENEOUS DENSITY AND $\sigma$ -DENSITY

We start this section with several definitions, and then establish the invariant property of uniform perfectness with respect to quasisymmetric maps in quasi-metric spaces (Lemma 3.2 below). Based on this result, Theorem 1.1 will be proved.

#### 3.1. Homogeneous density and $\sigma$ -density.

**Definition 3.1.** Suppose  $\{x_i\}_{i \in \mathbb{Z}}$  denotes a sequence of points in a quasi-metric space  $(Z, \rho)$  with  $a \neq x_i \neq b$ .

- (1) If  $x_i \rightarrow a$  as  $i \rightarrow -\infty$  and  $x_i \rightarrow b$  as  $i \rightarrow +\infty$ , then  $\{x_i\}$  is called a *chain* joining  $a$  and  $b$ ; further, if there is a constant  $\sigma > 1$  such that for all  $i$ ,

$$|\log r(a, x_i, x_{i+1}, b)| \leq \log \sigma,$$

then  $\{x_i\}$  is called a  $\sigma$ -chain.

- (2)  $(Z, \rho)$  is said to be  $\sigma$ -dense ( $\sigma > 1$ ) if any pair of points in  $(Z, \rho)$  can be joined by a  $\sigma$ -chain.

We remark that (1) a  $\sigma$ -dense space does not contain any isolated point and (2) a  $\sigma$ -dense space is  $\sigma'$ -dense for any  $\sigma' \geq \sigma$ .

**Definition 3.2.** A quasi-metric space  $(Z, \rho)$  is said to be *homogeneously dense*, abbreviated *HD*, if there are constants  $\lambda_1$  and  $\lambda_2$  with  $0 < \lambda_1 \leq \lambda_2 < 1$  such that for each pair of points  $a, b \in Z$ , there is  $x \in Z$  satisfying

$$\lambda_1 \rho(a, b) \leq \rho(a, x) \leq \lambda_2 \rho(a, b).$$

To emphasize the parameters, we also say that  $(Z, \rho)$  is  $(\lambda_1, \lambda_2)$ -HD.

**Lemma 3.1.** (1) If a quasi-metric space is  $(\lambda_1, \lambda_2)$ -HD, then it is  $(\lambda_1^n, \lambda_2^n)$ -HD for any  $n \in \mathbb{N}^+ = \{1, 2, \dots\}$ .

(2) Suppose that both  $(Z_1, \rho_1)$  and  $(Z_2, \rho_2)$  are quasi-metric spaces and that  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  is  $\eta$ -quasisymmetric. If  $(Z_1, \rho_1)$  is  $(\lambda_1, \lambda_2)$ -HD, then  $(Z_2, \rho_2)$  is  $(\mu_1, \mu_2)$ -HD, where both  $\mu_1$  and  $\mu_2$  depend only on  $\lambda_1, \lambda_2$  and  $\eta$ .

We remark that, in the setting of metric spaces, Lemma 3.1 coincides with [26, Lemma 3.9]. Also the proof of Lemma 3.1 is similar to that of [26, Lemma 3.9]. We omit it here.

**3.2. The invariant property of uniform perfectness with respect to quasisymmetric maps.** It is known that uniform perfectness is an invariant with respect to quasisymmetric maps in metric spaces (cf. [11, Exercise 11.2]). In the following, we prove that this fact is still valid in quasi-metric spaces.

**Lemma 3.2.** Let  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  be  $\eta$ -quasisymmetric, where both  $(Z_i, \rho_i)$  ( $i = 1, 2$ ) are quasi-metric. Then  $(Z_1, \rho_1)$  is uniformly perfect if and only if  $(Z_2, \rho_2)$  is uniformly perfect, quantitatively.

*Proof.* Since the inverse of a quasisymmetric map is also quasisymmetric, to prove the lemma, it suffices to show that the uniform perfectness of  $(Z_1, \rho_1)$  implies the uniform perfectness of  $(Z_2, \rho_2)$ . Now, we assume that  $(Z_1, \rho_1)$  is  $\mu$ -uniformly perfect



for some  $\mu \in (0, 1)$ . Then we shall show that  $(Z_2, \rho_2)$  is uniformly perfect too. To reach this goal, it suffices to find a constant  $\mu' \in (0, 1)$  such that for any  $z' \in Z_2$  and  $r > 0$ , if  $Z_2 \setminus \mathbb{B}(z', r) \neq \emptyset$ , then there is a point  $u'$  in  $(Z_2, \rho_2)$  such that

$$\mu' r \leq \rho_2(z', u') < r.$$

By the assumption  $Z_2 \setminus \mathbb{B}(z', r) \neq \emptyset$ , we see that there is a point  $u'_0 \in Z_2$  such that

$$(3.1) \quad \rho_2(z', u'_0) \geq r.$$

Choose  $0 < \alpha < 1$  small enough such that  $\eta(\alpha) < 1$ . Then there exists an integer  $k$  such that

$$(3.2) \quad \eta(\alpha)^k \rho_2(z', u'_0) < r \leq \eta(\alpha)^{k-1} \rho_2(z', u'_0).$$

Since  $(Z_1, \rho_1)$  is  $\mu$ -uniformly perfect and  $u_0 \in Z_1 \setminus \mathbb{B}(z, \alpha \rho_1(z, u_0))$ , we see that  $\mathbb{B}(z, \alpha \rho_1(z, u_0)) \setminus \mathbb{B}(z, \mu \alpha \rho_1(z, u_0)) \neq \emptyset$ . So there is a point  $u_1 \in Z_1$  such that

$$\mu \alpha \rho_1(z, u_0) \leq \rho_1(z, u_1) < \alpha \rho_1(z, u_0).$$

Hence

$$(3.3) \quad \mu' \rho_2(z', u'_0) \leq \rho_2(z', u'_1) < \eta(\alpha) \rho_2(z', u'_0),$$

where  $\mu' = \frac{1}{\eta(\frac{1}{\mu\alpha})}$ .

If  $\rho_2(z', u'_1) < r$ , then (3.1) and (3.3) lead to

$$\mu' r \leq \rho_2(z', u'_1) < r.$$

At present, we can take  $u' = u'_1$ .

Now, we consider the case:

$$(3.4) \quad \rho_2(z', u'_1) \geq r.$$

Since  $(Z_1, \rho_1)$  is  $\mu$ -uniformly perfect and  $u_1 \in Z_1 \setminus \mathbb{B}(z, \alpha \rho_1(z, u_1))$ , we see that  $\mathbb{B}(z, \alpha \rho_1(z, u_1)) \setminus \mathbb{B}(z, \mu \alpha \rho_1(z, u_1)) \neq \emptyset$ . So there is a point  $u_2 \in Z_1$  such that

$$\mu \alpha \rho_1(z, u_1) \leq \rho_1(z, u_2) < \alpha \rho_1(z, u_1).$$

Hence

$$\mu' \rho_2(z', u'_1) \leq \rho_2(z', u'_2) < \eta(\alpha) \rho_2(z', u'_1) < \eta(\alpha)^2 \rho_2(z', u'_0).$$

If  $\rho_2(z', u'_2) < r$ , then (3.4) leads to

$$\mu' r \leq \rho_2(z', u'_2) < r.$$

Hence, we can take  $u' = u'_2$ .

Next, we consider the case:

$$\rho_2(z', u'_2) \geq r.$$

By repeating this procedure, we can reach the following conclusion: There is  $u'_k \in Z_2$  such that

- (1) For any  $i \in \{1, \dots, k-1\}$ ,  $\rho_2(z', u'_i) \geq r$ ;
- (2)  $\mu' \rho_2(z', u'_{k-1}) \leq \rho_2(z', u'_k) < \eta(\alpha)^k \rho_2(z', u'_0)$ .

Then (3.2) guarantees that

$$\mu' r \leq \rho_2(z', u'_k) < r.$$

By taking  $u' = u'_k$ , we finish the proof.  $\square$

**3.3. The proof of Theorem 1.1.** By applying Lemmas 1.1, 3.1 and 3.2, together with Lemma 4.1 below, we see that the equivalence between (1) and (2) (resp. between (2) and (3)) easily follows from [11, Lemma 11.7] (resp. [14, Lemma 3.1]). Hence, to finish the proof, it remains to show the equivalence between (3) and (4), whose proof is as follows.

(3)  $\Rightarrow$  (4) Assume that  $(Z, \rho)$  is  $\sigma$ -dense. Let  $a, c, d$  be three distinct points in  $(Z, \rho)$ . Then there is a  $\sigma$ -chain  $\{x_i\}_{i \in \mathbb{Z}}$  in  $(Z, \rho)$  joining  $a$  and  $d$  such that

$$(3.5) \quad \frac{1}{\sigma} \leq r(a, x_i, x_{i+1}, d) \leq \sigma.$$

To prove this implication, it suffices to show that there is an integer  $k$  such that

$$(3.6) \quad \frac{1}{2\sigma^2} \leq r(a, x_{k-1}, c, d) \leq \frac{1}{2\sigma}.$$

For the proof, we let

$$k = \inf\{i \in \mathbb{Z} : r(a, x_i, c, d) < \frac{1}{2\sigma^2}\}.$$

Since  $\lim_{i \rightarrow +\infty} r(a, x_i, c, d) = 0$  and  $\lim_{i \rightarrow -\infty} r(a, x_i, c, d) = +\infty$ , we see that  $k$  is finite and so

$$r(a, x_k, c, d) < \frac{1}{2\sigma^2} \quad \text{and} \quad r(a, x_{k-1}, c, d) \geq \frac{1}{2\sigma^2}.$$

Then (3.5) implies that

$$r(a, x_{k-1}, c, d) = r(a, x_k, c, d)r(a, x_{k-1}, x_k, d) < \frac{1}{2\sigma}.$$

Hence (3.6) is true, and thus the implication from (3) to (4) is proved.

(4)  $\Rightarrow$  (3) For any two distinct points  $a$  and  $d \in Z$ , let  $c$  be a fixed point in  $(Z, \rho)$ , which is different from  $a$  and  $d$ . Then there is a point  $x_0 \in Z$  such that

$$\mu_1 \leq r(a, x_0, c, d) \leq \mu_2,$$

where  $0 < \mu_1 \leq \mu_2 < 1$ .

By repeating this procedure, we can find a sequence  $\{x_i\}_{i \in \mathbb{N}^+}$  in  $(Z, \rho)$  such that

$$\mu_1 \leq r(a, x_i, x_{i-1}, d) \leq \mu_2.$$

Then

$$\mu_1 \leq \frac{r(a, x_i, c, d)}{r(a, x_{i-1}, c, d)} = r(a, x_i, x_{i-1}, d) \leq \mu_2,$$

which implies that

$$\mu_1^{i+1} \leq r(a, x_i, c, d) \leq \mu_2^{i+1},$$

and so  $x_i \rightarrow d$  as  $i \rightarrow +\infty$ .

Similarly, we know that there exists  $\{x_{-i}\}_{i \in \mathbb{N}_+}$  in  $(Z, \rho)$  such that

$$\mu_1 \leq r(d, x_{-i}, x_{1-i}, a) = r(a, x_{1-i}, x_{-i}, d) \leq \mu_2$$

and

$$\mu_1 \mu_2^{1-i} \leq r(a, x_{-i}, c, d) \leq \mu_1^{1-i} \mu_2.$$

Then  $x_{-i} \rightarrow a$  as  $i \rightarrow +\infty$ , and hence we have proved that  $(Z, \rho)$  is  $\frac{1}{\mu_1}$ -dense.  $\square$

#### 4. THE INVARIANT PROPERTY OF UNIFORM PERFECTNESS WITH RESPECT TO QUASIMÖBIUS MAPS

The aim of this section is to prove Theorem 1.2. To this end, by Theorem 1.1, it suffices to show the following lemma.

**Lemma 4.1.** *Let  $f : (Z_1, \rho_1) \rightarrow (Z_2, \rho_2)$  be  $\theta$ -quasimöbius, where both  $(Z_i, \rho_i)$  ( $i = 1, 2$ ) are quasi-metric. Then  $(Z_1, \rho_1)$  is  $\sigma$ -dense if and only if  $(Z_2, \rho_2)$  is  $\sigma'$ -dense, quantitatively.*

*Proof.* Since the inverse of a  $\theta$ -quasimöbius map is  $\theta'$ -quasimöbius with  $\theta'(t) = \frac{1}{\theta-1(1/t)}$ , it suffices to show that  $(Z_2, \rho_2)$  is  $\sigma'$ -dense under the assumption that  $(Z_1, \rho_1)$  is  $\sigma$ -dense, where  $\sigma > 1$  and  $\sigma'$  depends only on  $\sigma$  and  $\theta$ . Obviously, we only need to check that for each pair of points  $a', b'$  in  $(Z_2, \rho_2)$ , there is a  $\sigma'$ -chain in  $(Z_2, \rho_2)$  joining them. Now, we assume that  $\{x_i\}_{i \in \mathbb{Z}}$  is a  $\sigma$ -chain in  $(Z_1, \rho_1)$  joining the points  $a$  and  $b$  with

$$\frac{1}{\sigma} \leq r(a, x_i, x_{i+1}, b) \leq \sigma.$$

Then for all  $i$ , we have

$$\frac{1}{\theta(\sigma) + 1} \leq r(a', x'_i, x'_{i+1}, b') \leq \theta(\sigma) + 1,$$

which shows that  $\{x'_i\}_{i \in \mathbb{Z}}$  is a  $\sigma'$ -chain in  $(Z_2, \rho_2)$  joining  $a'$  and  $b'$  with  $\sigma' = \theta(\sigma) + 1$ .  $\square$

#### 5. UNIFORM PERFECTNESS, (POWER) QUASISYMMETRIC MAPS AND (POWER) QUASIMÖBIUS MAPS

This section is devoted to the proof of Theorem 1.3 concerning the relationships among uniform perfectness, (power) quasisymmetric maps and (power) quasimöbius maps in quasi-metric spaces. It consists of two subsections. In the first subsection, we shall prove a relationship among uniform perfectness, quasisymmetric maps and power quasisymmetric maps, i.e. Theorem 1.3(1), and in the second subsection, the proof of a relationship among uniform perfectness, quasimöbius maps and power quasimöbius maps, i.e. Theorem 1.3(2), will be presented.

**5.1. The proof of Theorem 1.3(1).** It follows from Lemma 1.1 that there exist a constant  $\varepsilon > 0$  with  $K^\varepsilon \leq 2$  and a metric  $d_\varepsilon$  (briefly  $d$  in the following) in  $Z$  such that

$$\frac{1}{4} \rho^\varepsilon(z_1, z_2) \leq d(z_1, z_2) \leq \rho^\varepsilon(z_1, z_2)$$

for all  $z_1, z_2 \in Z$ . Let  $id$  denote the identity map from  $(Z, \rho)$  to  $(Z, d)$ , i.e.,

$$id : (Z, \rho) \rightarrow (Z, d).$$

Obviously,  $id$  is power quasisymmetric with its control function  $\eta(t) = 4(t^\epsilon \vee t^{\frac{1}{\epsilon}})$ .

We first assume that  $(Z, \rho)$  is uniformly perfect, and consider a quasisymmetric map  $f$  defined in  $(Z, \rho)$ . It follows from the power quasisymmetry of  $id$  and Lemma 3.2 that  $(Z, d)$  is uniformly perfect, and so Theorem 1.1 implies that  $(Z, d)$  is  $(\lambda_1, \lambda_2)$ -HD for constants  $\lambda_1$  and  $\lambda_2$  with  $0 < \lambda_1 \leq \lambda_2 < 1$ . Since  $f \circ id^{-1}$  is quasisymmetric in  $(Z, d)$ , we see from [26, Corollary 3.12] that  $f \circ id^{-1}$  is power quasisymmetric, which implies that  $f$  itself is power quasisymmetric.

Next, we assume that every quasisymmetric map of  $(Z, \rho)$  is power quasisymmetric. Then we see that for any quasisymmetric map  $g$  in  $(Z, d)$ ,  $g \circ id$  is power quasisymmetric in  $(Z, \rho)$ , and so  $g$  itself is power quasisymmetric. Hence, by [25, Theorems 4.13 and 6.20],  $(Z, d)$  is uniformly perfect. Since  $id$  is power quasisymmetric, it follows from Lemma 3.2 that  $(Z, \rho)$  is uniformly perfect.  $\square$

**5.2. The proof of Theorem 1.3(2).** We start this subsection with the following result in metric spaces.

**Lemma 5.1.** *Suppose  $(Z, d)$  is a metric space with no isolated points. Then the following statements are quantitatively equivalent.*

- (1)  $(Z, d)$  is uniformly perfect;
- (2) every quasimöbius map of  $(Z, d)$  is power quasimöbius.

*Proof.* By [1, Theorem 3.2], we only need to prove the implication from (2) to (1). Assume that every quasimöbius map in  $(Z, d)$  is a power quasimöbius map. To prove the uniform perfectness of  $(Z, d)$ , we divide the proof into two cases.

**Case 5.1.**  $(Z, d)$  is unbounded.

We shall apply Theorem 1.3(1) to finish the proof in this case. For this, we assume that  $f$  is a quasisymmetric map in  $(Z, d)$ . Then Lemma 2.2 implies that  $f$  is quasimöbius, and further, [29, Theorem 3.10] guarantees that  $f(z) \rightarrow \infty$  as  $z \rightarrow \infty$ . Again, it follows from [29, Theorem 3.10] that  $f$  is power quasisymmetric, and so Theorem 1.3(1) guarantees that  $(Z, d)$  is uniformly perfect. Hence the lemma is true in this case.

**Case 5.2.**  $(Z, d)$  is bounded.

By the Kuratowski embedding theorem [17], we may assume that  $Z$  is a subset of a Banach space  $E$ . By performing an auxiliary translation, further, we assume that  $0 \in Z$ . Let

$$u(x) = \frac{x}{|x|^2}$$

be the inversion in  $\dot{E} = E \cup \{\infty\}$ . Then, clearly,  $u(Z)$  is unbounded. By [29, §1.6],  $u$  is  $\theta$ -quasimöbius, where  $\theta(t) = 81t$ , and obviously, it is power quasimöbius. To prove that  $(Z, d)$  is uniformly perfect, by Theorem 1.2, it suffices to show that  $u(Z)$  is uniformly perfect. Again, we shall apply Theorem 1.3(1) to reach this goal. For this, we assume that  $g$  is quasisymmetric in  $u(Z)$ . Once more, by Lemma 2.2,  $g$  is quasimöbius. Then  $g \circ u$  is quasimöbius in  $(Z, d)$ , which implies that  $g \circ u$  is power quasimöbius, and thus Lemma 2.3(3) guarantees that  $g$  itself is power quasimöbius.

So [29, Theorem 3.10] guarantees that  $g$  is power quasisymmetric. Then it follows from Theorem 1.3(1) that  $u(Z)$  is uniformly perfect. Hence the proof of the lemma is complete.  $\square$

**The proof of Theorem 1.3(2).** Let  $id : (Z, \rho) \rightarrow (Z, d)$  be the same as that in the proof of Theorem 1.3(1). Then  $id$  is power quasisymmetric with its control function  $\eta(t) = 4(t^\epsilon \vee t^{\frac{1}{\epsilon}})$  and  $\epsilon \in (0, 1)$ .

Assume now that  $(Z, \rho)$  is uniformly perfect, and so is  $(Z, d)$  by Lemma 3.2. For any quasimöbius map  $f$  in  $(Z, \rho)$ , it follows from Lemma 5.1 that  $f \circ id$  is power quasimöbius, and so is  $f$  itself by Lemma 2.3. This shows that the necessity in Theorem 1.3(2) is true.

To prove the sufficiency in Theorem 1.3(2), it suffices to prove the uniform perfectness of  $(Z, d)$  under the assumption that every quasisymmetric map in  $(Z, \rho)$  is power quasimöbius. By Lemma 5.1, we only need to show the power quasisymmetry of each quasisymmetric map in  $(Z, d)$ . This fact easily follows from Lemma 2.3. Hence the proof of Theorem 1.3(2) is complete.  $\square$

## 6. APPLICATIONS

The aim of this section is twofold. First, as an application of Theorem 1.2, we will give a different proof to [20, Theorem 7.1]. Second, we shall apply Theorem 1.2 to discuss the uniform perfectness of a complete quasi-metric space and the corresponding boundary of its hyperbolic approximation.

**6.1. Application I.** We begin this subsection with a definition.

**Definition 6.1.** For  $p \in (Z, \rho)$ , let

$$\rho_p(x, y) = \frac{r^2 \rho(x, y)}{\rho(x, p) \rho(y, p)}$$

for all  $x, y \in Z \setminus \{p\}$ . Then  $\rho_p$  is said to be *the inversion* with respect to  $\rho$  centered at  $p$  with radius  $r > 0$ .

**Theorem 6.1.** ([20, Theorem 7.1]) *For any  $p \in Z$ , if  $(Z \setminus \{p\}, \rho)$  is a uniformly perfect quasi-metric space, then  $(Z \setminus \{p\}, \rho_p)$  is a uniformly perfect quasi-metric space.*

*Proof.* First, if  $(Z \setminus \{p\}, \rho)$  is a  $K$ -quasi-metric space, by [7, Proposition 5.3.6], we know that  $(Z \setminus \{p\}, \rho_p)$  is a  $K^2$ -quasi-metric space. Then a direct computation gives that the identity map from  $(Z \setminus \{p\}, \rho)$  to  $(Z \setminus \{p\}, \rho_p)$  is  $\theta$ -quasimöbius with  $\theta(t) = t$ . Hence the proof of the theorem easily follows from Theorem 1.2.  $\square$

**6.2. Application II.** Let  $Hyp_r(Z, \rho)$  denote the hyperbolic approximation of  $(Z, \rho)$  with parameter  $r$ ,  $\partial_\infty^{a,o} Hyp_r(Z, \rho)$  the boundary at infinity of  $Hyp_r(Z, \rho)$  with respect to the quasi-metric  $a^{-(\cdot|\cdot)_o}$  based at  $o \in Hyp_r(Z, \rho)$  with  $a > 1$ , and  $\partial_\infty^{a',b} Hyp_r(Z, \rho)$  the boundary at infinity of  $Hyp_r(Z, \rho)$  with respect to the quasi-metric  $a'^{-(\cdot|\cdot)_\omega}$  based at  $\omega$  with  $a' > 1$ , where  $b$  is a Busemann function based at  $\omega$ . See [16, §3] for their precise definitions.

**Theorem 6.2.** *Suppose  $(Z, \rho)$  is a complete quasi-metric space and  $r \in (0, 1)$ . Then the following are quantitatively equivalent.*

- (1)  $(Z, \rho)$  is uniformly perfect;
- (2)  $\partial_{\infty}^{a,o} \text{Hyp}_r(Z, \rho)$  is uniformly perfect;
- (3)  $\partial_{\infty}^{a,b} \text{Hyp}_r(Z, \rho)$  is uniformly perfect.

*Proof.* First, by [7, Proposition 2.2.9 and 5.2.8], we know that the identity map from  $\partial_{\infty}^{a,b} \text{Hyp}_r(Z, \rho)$  to  $\partial_{\infty}^{a,o} \text{Hyp}_r(Z, \rho)$  is quasimöbius, and so Theorem 1.2 implies the quantitative equivalence of (2) and (3).

To finish the proof of the theorem, we divide the discussions into two cases. The first case is that  $(Z, \rho)$  is unbounded. By [16, Theorem 3], we know that for any Busemann function  $b \in B(\omega)$ , the identity map from  $\partial_{\infty}^{a,b} \text{Hyp}_r(Z, \rho)$  to  $(Z, \rho)$  is bi-Hölder, and thus Theorem 1.2 guarantees the quantitative equivalence of (1) and (3). For the remainder case, that is,  $(Z, \rho)$  is bounded, again, by [16, Theorem 3], we see that the identity map from  $\partial_{\infty}^{a,o} \text{Hyp}_r(Z, \rho)$  to  $(Z, \rho)$  is bi-Hölder. Once more, it follows from Theorem 1.2 that (1) and (2) are quantitatively equivalent. Hence the proof of this theorem is complete.  $\square$

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